# Topological solitons: a short course for undergraduates

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## Rough plan

- Monday
  - Session 1: Lecture "Kinks"
  - Session 2: Lecture "Lagrangian field theory"
  - Session 3: Lecture "Higher dimensions"
  - Session 4: Problems class
- Tuesday
  - Session 5: Lecture "Lumps"
  - Session 6: Lecture "The geodesic approximation"

## Exercises for session 4

1. Show that any field theory with Lagrangian density of the form $\mathcal{L}(\phi_t, \phi_x, \phi)$ conserves momentum

$$P = -\int_{-\infty}^{\infty} \frac{\partial \mathcal{L}}{\partial \phi_t} \phi_x dx.$$

   Verify that this reproduces the claimed conserved momentum for the sine-Gordon model.

2. Consider the $\phi^4$ model:

$$\mathcal{L} = \frac{1}{2}(\phi_t^2 - \phi_x^2) - \frac{1}{2}(1 - \phi^2)^2.$$

   (a) Compute its conserved energy $E$.

   (b) Show that any static field $\phi : \mathbb{R} \to \mathbb{R}$ with kink boundary conditions ($\lim_{x\to\pm\infty} \phi(x) = \pm 1$) has $E \geq \frac{4}{3}$. Construct all static fields that attain this bound. (Hint: repeat the Bogomol'nyi trick.)

3. Consider
$$n(\phi) = \frac{1}{4\pi}\int_{\mathbb{R}^2} \phi\cdot(\phi_x \times \phi_y)dxdy$$
as a functional of the field $\phi : \mathbb{R}^2 \to S^2$. Show that this is a topological invariant of $\phi$. That is, show that, for any smooth variation $\phi_s$ of $\phi = \phi_0$ of compact support,
$$\left.\frac{dn(\phi_s)}{ds}\right|_{s=0} = 0.$$

4. Let $\phi : \mathbb{R}^2 \to S^2$ be a finite energy static solution of the field theory with static energy functional
$$E(\phi) = \int_{\mathbb{R}^2}\left\{\frac{1}{2}(|\phi_x|^2 + |\phi_y|^2) + \frac{1}{4}|\phi_x \times \phi_y|^2 + (1-\phi_3)\right\}dxdy.$$
Such solutions are called *baby Skyrmions*. Show that
$$\int_{\mathbb{R}^2}|\phi_x|^2dxdy = \int_{\mathbb{R}^2}|\phi_y|^2dxdy$$
and
$$\int_{\mathbb{R}^2}\frac{1}{4}|\phi_x\times\phi_y|^2dxdy = \int_{\mathbb{R}^2}(1-\phi_3)dxdy.$$
(Hint: The second should come from Derrick's scaling argument. Can you modify this argument to get the first?)

## Solutions

1. For an aribtrary field $\phi : \mathbb{R}^{1,1} \to \mathbb{R}$ consider the variation of $\phi$ generating spatial translations
$$\phi_s(t,x) = \phi(t, x-s).$$
Then $\mathcal{L}_{\phi_s}(t,x) = \mathcal{L}_\phi(t,x-s)$, since $\mathcal{L}$ does not depend on $x$. Hence
$$\partial_s|_{s=0}\mathcal{L}_{\phi_s} = -\partial_x\mathcal{L}_\phi. \tag{1}$$
But $\mathcal{L}_{\phi_s}(t,x) = \mathcal{L}(\phi_{s,t}(t,x), \phi_{s,x}(t,x), \phi_s(t,x))$, so
$$\begin{aligned}
\partial_s|_{s=0}\mathcal{L}_{\phi_s} &= \frac{\partial\mathcal{L}}{\partial\phi_t}\partial_s|_{s=0}\phi_{s,t} + \frac{\partial\mathcal{L}}{\partial\phi_x}\partial_s|_{s=0}\phi_{s,x} + \frac{\partial\mathcal{L}}{\partial\phi}\partial_s|_{s=0}\phi_s \\
&= -\frac{\partial\mathcal{L}}{\partial\phi_t}\phi_{xt} - \frac{\partial\mathcal{L}}{\partial\phi_x}\phi_{xx} - \frac{\partial\mathcal{L}}{\partial\phi}\phi_x \\
&= -\partial_t\left(\frac{\partial\mathcal{L}}{\partial\phi_t}\phi_x\right) + \partial_t\left(\frac{\partial\mathcal{L}}{\partial\phi_t}\right)\phi_x - \partial_x\left(\frac{\partial\mathcal{L}}{\partial\phi_x}\phi_x\right) + \partial_x\left(\frac{\partial\mathcal{L}}{\partial\phi_x}\right)\phi_x - \frac{\partial\mathcal{L}}{\partial\phi}\phi_x \\
&= -\phi_x\left(\partial_t\left(\frac{\partial\mathcal{L}}{\partial\phi_t}\right) + \partial_x\left(\frac{\partial\mathcal{L}}{\partial\phi_x}\right) + \frac{\partial\mathcal{L}}{\partial\phi}\right) - \partial_t\left(\frac{\partial\mathcal{L}}{\partial\phi_t}\phi_x\right) - \partial_x\left(\frac{\partial\mathcal{L}}{\partial\phi_x}\phi_x\right).
\end{aligned} \tag{2}$$

Hence, if $\phi$ is a solution of the field theory (so satisfies the Euler-Lagrange equation),

$$\partial_s|_{s=0}\mathcal{L}_{\phi_s} = -\partial_t\left(\frac{\partial\mathcal{L}}{\partial\phi_t}\phi_x\right) - \partial_x\left(\frac{\partial\mathcal{L}}{\partial\phi_x}\phi_x\right). \tag{3}$$

Equating (1) and (3), we see that, for all solutions $\phi(t,x)$,

$$\partial_t J^0 + \partial_x J^1 = 0$$

where

$$\begin{aligned} J^0 &= -\frac{\partial\mathcal{L}}{\partial\phi_t}\phi_x \\ J^1 &= -\frac{\partial\mathcal{L}}{\partial\phi_x}\phi_x + \mathcal{L}_\phi. \end{aligned}$$

It follows that

$$P(t) = \int_{-\infty}^{\infty} J^0(t,x)dx = -\int_{-\infty}^{\infty}\frac{\partial\mathcal{L}}{\partial\phi_t}\phi_x dx$$

is a conserved quantity, since

$$\frac{dP}{dt} = -\int_{-\infty}^{\infty}\partial_x J^1 dx = J^1(-\infty) - J^1(\infty) = 0$$

if we assume sensible boundary conditions.

In the case of the sine-Gordon model,

$$\mathcal{L} = \frac{1}{2}phi_t^2 - \frac{1}{2}\phi_x^2 - (1-\cos\phi)$$

so

$$P = -\int_{-\infty}^{\infty}\phi_t\phi_x dx$$

as claimed in Lecture 1.

2. (a) Using the formula from Lecture 2,

$$E = \int_{-\infty}^{\infty}\left(\frac{\partial\mathcal{L}}{\partial\phi_t}\phi_t - \mathcal{L}_\phi\right)dx = \frac{1}{2}\int_{-\infty}^{\infty}\left(\phi_t^2 + \phi_x^2 + (1-\phi^2)^2\right)dx.$$

(b) Assume now that $\phi : \mathbb{R} \to \mathbb{R}$ is a static field with boundary values $\phi(\pm\infty) = \pm 1$. Then

$$\begin{aligned} 0 &\leq \frac{1}{2}\int_{\mathbb{R}}\left(\phi_x - (1-\phi^2)\right)^2 dx \\ &= E(\phi) - \int_{\mathbb{R}}\phi_x(1-\phi^2)dx \\ &= E(\phi) - \left[\phi(x) - \frac{\phi(x)^3}{3}\right]_{-\infty}^{\infty} \\ &= E(\phi) - \frac{4}{3}. \end{aligned}$$

Hence, $E(\phi) \geq \frac{4}{3}$, with equality if and only if

$$\phi_x = (1 - \phi^2)$$

whose general solution is $\phi(x) = \tanh(x - x_0)$.

3. Let $\phi_s$ be a smooth variation of $\phi = \phi_0$ with compact support $\Omega \subset \mathbb{R}^2$ (meaning that, for all $(x, y) \notin \Omega$, $\phi_s(x, y) = \phi(xy)$ for all $s$), and denote by $\varepsilon : \mathbb{R}^2 \to \mathbb{R}^3$

$$\varepsilon(x, y) = \partial_s \phi_s(x, y)|_{s=0}.$$

Note that $\varepsilon = 0$ outside $\Omega$ and on $\partial\Omega$, and that, for all $(x, y) \in \mathbb{R}^2$,

$$\varepsilon(x, y) \cdot \phi(x, y) = 0,$$

that is, $\varepsilon(x, y) \in T_{\phi(x,y)} S^2$. (This follows by differentiating the identity $|\phi_s(x, y)|^2 = 1$ with respect to $s$.)

Consider the corresponding variation of $n$:

$$\left.\frac{dn(\phi_s)}{ds}\right|_{s=0} = \int_\Omega (\varepsilon \cdot (\phi_x \times \phi_y) + \phi \cdot (\varepsilon_x \times \phi_y) + \phi \cdot (\phi_x \times \varepsilon_y))\, dxdy.$$

Note that $\phi_x, \phi_y, \varepsilon$ are all in the two-dimensional space orthogonal to $\phi$, so the first term vanishes identically (it is the determinant of the $3 \times 3$ matrix whose rows are those 3 vectors, which must be linearly dependent). To handle the second and third terms, we pull our usual trick:

$$\begin{aligned} \left.\frac{dn(\phi_s)}{ds}\right|_{s=0} &= \int_\Omega \Big(\partial_x(\phi \cdot (\varepsilon \times \phi_y)) - \phi_x \cdot (\varepsilon \times \phi_y) - \phi \cdot (\varepsilon \times \phi_{xy}) \\ &\qquad + \partial_y(\phi \cdot (\phi_x \times \varepsilon)) - \phi_y \cdot (\phi_x \times \varepsilon) - \phi \cdot (\phi_{xy} \times \varepsilon)\Big) dxdy. \end{aligned}$$

The second and fifth terms vanish identically (since $\phi_x, \phi_y, \varepsilon$ are all in the plane orthogonal to $\phi$), and the sixth cancels the third. Hence

$$\begin{aligned} \left.\frac{dn(\phi_s)}{ds}\right|_{s=0} &= \int_\Omega \Big(\partial_x(\phi \cdot (\varepsilon \times \phi_y)) + \partial_y(\phi \cdot (\phi_x \times \varepsilon)\Big) dxdy \\ &= \int_{\partial\Omega} \mathbf{A} \cdot d\mathbf{s} \end{aligned}$$

the flux of the vector field

$$\mathbf{A} = (\phi \cdot (\varepsilon \times \phi_y), \phi \cdot (\phi_x \times \varepsilon))$$

through the boundary $\partial\Omega$. But $\varepsilon$ vanishes on $\partial\Omega$, so $\mathbf{A} = 0$ on $\partial\Omega$, and hence

$$\left.\frac{dn(\phi_s)}{ds}\right|_{s=0} = 0.$$

Since this hold for all variations, $n$ is a topological invariant of the field $\phi$.

4. Define the terms

$$\begin{aligned} E_0(\phi) &= \int_{\mathbb{R}^2}(1-\phi_3)dxdy \\ E_2(\phi) &= \frac{1}{2}\int_{\mathbb{R}^2}(|\phi_x|^2+|\phi_y|^2)dxdy \\ E_4(\phi) &= \frac{1}{2}\int_{\mathbb{R}^2}|\phi_x\times\phi_y|^2dxdy \end{aligned}$$

so that $E = E_2 + E_4 + E_0$. (The subscript denotes the degree of the integrand regarded as a polynomial in the spatial derivatives $\phi_x, \phi_y$.) Consider the behaviour of these functionals under the scaling variation

$$\phi_\lambda(x,y) = \phi(\lambda x \lambda y).$$

$$\begin{aligned} E_0(\phi_\lambda) &= \int_{\mathbb{R}^2}(1-\phi_3(\lambda x\lambda y))dxdy = \lambda^{-2}E_0(\phi) \\ E_2(\phi) &= \frac{1}{2}\int_{\mathbb{R}^2}\lambda^2(|\phi_x(\lambda x,\lambda y)|^2+|\phi_y(\lambda x,\lambda y)|^2)dxdy = E_2(\phi) \\ E_4(\phi) &= \frac{1}{2}\int_{\mathbb{R}^2}\lambda^4|\phi_x(\lambda x,\lambda y)\times\phi_y(\lambda x,\lambda y)|^2dxdy = \lambda^2E_4(\phi) \end{aligned}$$

where, in each case the final equality follows from changing to rescaled coordinates $(x', y') = (\lambda x, \lambda y)$ in the integral. So

$$E(\phi_\lambda) = E_2(\phi) + \lambda^2E_4(\phi) + \lambda^{-2}E_0(\phi).$$

Since $\phi$ is a static solution, $E$ is stationary with respect to all variations of $\phi$, including this scaling variation, so

$$\left.\frac{dE(\phi_\lambda)}{d\lambda}\right|_{\lambda=1} = 0,$$

whence we see that $2E_4(\phi) - 2E_0(\phi) = 0$, the second claimed identity.

To get the first claimed identity, we use instead the variation

$$\phi_\lambda(x,y) = \phi(\lambda x, \lambda^{-1}y).$$

$$\begin{aligned} E_0(\phi_\lambda) &= \int_{\mathbb{R}^2}(1-\phi_3(\lambda x,\lambda^{-1}y))dxdy = E_0(\phi) \\ E_2(\phi) &= \frac{1}{2}\int_{\mathbb{R}^2}(\lambda^2|\phi_x(\lambda x,\lambda^{-1}y)|^2+\lambda^{-2}|\phi_y(\lambda x,\lambda^{-1}y)|^2)dxdy \\ &= \frac{\lambda^2}{2}\int_{\mathbb{R}^2}|\phi_x|^2dxdy + \frac{\lambda^{-2}}{2}\int_{\mathbb{R}^2}|\phi_y|^2dxdy \\ E_4(\phi) &= \frac{1}{2}\int_{\mathbb{R}^2}|\lambda\phi_x(\lambda x,\lambda^{-1}y)\times\lambda^{-1}\phi_y(\lambda x,\lambda^{-1}y)|^2dxdy = E_4(\phi) \end{aligned}$$

where, in each case the final equality follows from changing to rescaled coordinates $(x', y') = (\lambda x, \lambda^{-1} y)$ in the integral. (Note that this coordinate transformation is area preserving, that is, $dx'dy' = dxdy$). Again, $E$ must be stationary with respect to this variation, so

$$0 = \left.\frac{dE(\phi_\lambda)}{d\lambda}\right|_{\lambda=1} = \int_{\mathbb{R}^2} |\phi_x|^2 dxdy - \int_{\mathbb{R}^2} |\phi_y|^2 dxdy.$$

## Take home exercises

1. Compute $n(\phi)$, the degree of the map $\phi : \mathbb{R}^2 \to S^2$,

$$\phi(r, \theta) = (\sin f(r) \cos k\theta, sin f(r) \sin k\theta, \cos f(r)),$$

where $f : [0, \infty) \to \mathbb{R}$ is a strictly decreasing function with $f(0) = \pi$ and $\lim_{r\to\infty} f(r) = 0$ and $k$ is an integer. We are using plane polar coordinates here (i.e. $(x, y) = r(\cos\theta, \sin\theta)$).

The obvious strategy is to compute the integral formula for $n$. This is straightforward if you're familiar with some basic ideas about differential forms, but gets quite laborious if not. A much easier strategy is to count signed preimages. If you're really keen, compute it both ways and check you get the same answer.

2. Assign to each point $(\phi_1, \phi_2, \phi_3) \in S^2$ the stereographic coordinate

$$w = \frac{\phi_1 + i\phi_2}{1 + \phi_3}.$$

   (a) Compute the stereographic coordinates of $(1, 0, 0)$, $(-1, 0, 0)$, $(0, 1, 0)$, $(0, -1, 0)$, $(0, 0, 1)$, $(0, 0, -1)$, and mark these coordinates on a diagram of the sphere.

   (b) Show that

$$\phi = \frac{(2\mathrm{Re}w, 2\mathrm{Im}w, 1 - |w|^2)}{1 + |w|^2}.$$

   (c) Show that the Polyakov equation

$$\phi_x + \phi \times \phi_y = 0$$

   rewritten in coordinates $z = x + iy$ on $\mathbb{R}^2$ and $w$ on $S^2$ is

$$\frac{\partial w}{\partial \bar{z}} = 0,$$

   the Cauchy Riemann equation.

3. Let $\phi : \mathbb{R}^2 \to S^2$ be the general 1-lump solution, that is, in stereographic coordinates

$$w(z) = \frac{a_1}{z + b_1}, \qquad (a_1, b_1) \in (\mathbb{C}^\times \times \mathbb{C}.$$

Show that the energy density

$$\mathcal{E}(x, y) = \frac{1}{2}(|\phi_x|^2 + |\phi_y|^2)$$

of this solution is a lump centred at $z = -b_1$ with width $\sim |a_1|$.

4. Describe $\mathcal{E}$ for the "coincident" 2-lump

$$w(z) = \frac{a_2}{z^2}.$$

Is it just a bigger lump centred at $z = 0$?

5. Let $\phi$ be a 1-lump, that is

$$w(z) = \frac{q_1 + iq_2}{z + q_3 + iq_4}$$

for some $(q_1, q_2, q_3, q_4)$. Try to compute the metric coefficient $g_{11}$. What goes wrong?

6. Construct the most general $n$-lump

$$w(z) = \frac{a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n}{z^n + b_1 z^{n-1} + \cdots + b_n}$$

invariant under all rotations

$$w(z) \mapsto \alpha^{-n} w(\alpha z), \qquad \alpha \in \mathbb{C}, |\alpha| = 1.$$

You should find that the space of all such $n$-lumps forms a submanifold of $M_n$ diffeomorphic to $\mathbb{C}^\times$. Compute the induced metric $g$ on this submanifold. Analyze its geodesic flow.

## Further reading


1. N.S. Manton and P.M. Sutcliffe, *Topological Solitons*, Cambridge University Press, 2004. The bible of the subject.

2. Yisong Yang, *Solitons in Field Theory and Nonlinear Analysis*, Springer, 2001. Rigorously develops the underlying mathematical analysis.

3. M. Nakahara, *Geometry, Topology and Physics*, Taylor Francis, 1990. A nice introduction to pretty much all the differential geometry and topology used in theoretical physics.

4. N.S. Manton, "Solitons as elementary particles: a paradigm scrutinized," *Nonlinearity* **21** (2008) T221. A critical appraisal of the idea that elementary particles might really *be* topological solitons.